Integrable fourth-order difference equations

2010 J. Phys. A: Math. Theor. 43 235203

(http://iopscience.iop.org/1751-8121/43/23/235203)

J. Phys. A: Math. Theor. **43** (2010) 235203 (13pp) doi:10.1088/1751-8113/43/23/235203

Integrable fourth-order difference equations

C Uma Maheswari and R Sahadevan

Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai-600 005, Tamil Nadu, India

E-mail: umagenie2004@hotmail.com and ramajayamsaha@yahoo.co.in

Received 1 October 2009, in final form 9 April 2010
Published 13 May 2010
Online at stacks.iop.org/JPhysA/43/235203

Abstract
In this paper an attempt is made to find four-dimensional analogs of two-dimensional Quispel, Roberts and Thompson mappings and identified four distinct cases have been identified. The obtained mappings are measure preserving. The integrability of the isolated mappings is examined by constructing a sufficient number of integrals and their symplectic structure wherever possible.

PACS numbers: 02.30.Ik, 02.30.Jr, 45.20.Jj

1. Introduction

Given a nonlinear ordinary or partial differential equation, particularly the integrable equations, how to discretize it, preserving integrability, has been one of the topics of interest in recent years [1, 2, 4, 8, 14, 16–21]. Considerable progress has been made in this direction during the past 20 years or so and several integrable differential–difference, difference–difference (or lattice equations) and ordinary difference equations (or mappings) [5, 9–11, 22, 23, 27–29], including discrete Painlevé equations [7, 8], have been reported. In this paper we confine our attention to ordinary difference equations ($O\Delta E$). During the late 1980s, Quispel, Roberts and Thompson (QRT) [20, 21] reported an 18-parameter integrable mapping in the plane which can be viewed as a discrete version of a second-order autonomous nonlinear ordinary differential equation. This path-breaking result has led to the construction of the well-known discrete Painlevé equations. Systematic efforts to analyze a third-order autonomous ordinary nonlinear difference equation particularly from the point of view of integrability have been made by several researchers in recent years [3, 10, 12, 15, 19, 23, 25, 26]. For instance, Iatrou [12] has identified few third-order integrable difference equations which admit two independent polynomial integrals. Similarly, Matsukidaira and Takahashi [15] have shown how third-order integrable difference equations can be generated by a pair of second-order equations. In [26] we have reported several integrable three-dimensional analogs of QRT mappings. We mention here that one of the authors of this paper has considered in [6] a rational integral and identified six fourth-order autonomous mappings with two integrals, which can also be derived from the similarity reductions of discrete–discrete modified Korteweg de Vries and sine Gordon lattice equations. They have adopted the following strategy. First they have determined that under what parametric conditions the rational integral is cyclic, reversible and symplectic with a specific symplectic matrix. The second integral of the identified mappings has been derived from the reductions of the discrete–discrete modified Korteweg de Vries and sine-Gordon lattice equations. The purpose of this paper is to explain how to construct two or more integrals directly for fourth-order difference equations leading to integrable four-dimensional analogs of the two-dimensional QRT mappings [5, 6, 13, 24, 25].

It is appropriate to mention here that there exists no unique definition of integrability for nonlinear difference equations like for nonlinear differential equations. However, to investigate the integrability nature of nonlinear ordinary and partial difference equations there exist working definitions in the literature. We recall the following to understand the working definitions.

Consider an autonomous Nth-order ordinary difference equation

$$w_{n+N} = F(w_n, w_{n+1}, \ldots, w_{n+N-1}), \qquad w_{n+N} = w(n+N), \tag{1.1}$$

which can be rewritten as a system of first-order $O\Delta E$s

$$\vec{w}_{i+1} = \vec{G} \tag{1.2}$$

where $\vec{G} = (F_1, F_2, \ldots, F_N = F)$ and F_i's are functions of $(w_n, w_{n+1}, \ldots, w_{n+N-1})$.

Integral

An integral (also referred to as conserved quantity) for the $O\Delta E$ (1.1) is a function $I(n) = I(w_n, \ldots, w_{n+N-1})$ that is not identically constant but is constant on all solutions of the $O\Delta E$. That is, $I(n)$ is an integral for the $O\Delta E$ (1.1) if $I(w_n, \ldots, w_{n+N-1}) = I(w_{n+1}, \ldots, w_{n+N})$ holds.

Measure preserving map

A mapping $L : (w_1, \ldots, w_n) \rightarrow (\tilde{w}_1, \ldots, \tilde{w}_n)$ is said to be measure preserving with density $m(w_1, \ldots, w_n)$ if the Jacobian determinant $J(w_1, \ldots, w_n) = \det \mathrm{d}L(w_1, \ldots, w_n) = \pm m(w_1, \ldots, w_n)/m(\tilde{w}_1, \ldots, \tilde{w}_n)$.

Symplectic map

A mapping say $L : R^{2N} \rightarrow R^{2N}$ is said to be symplectic, if there exists an anti-symmetric $(2N \times 2N)$ matrix $\Omega(n)$ satisfying the following conditions.

(i) $J(n)\Omega(n)J(n)^T = \Omega(n+1)$
where $J(n)$ is the Jacobian of the mapping L.
(ii) $\Omega(n)$ has maximal rank.
(iii) Jacobi identity.

We focus our attention on the following working definition.

A $2N$th-order $O\Delta E$ (1.1) is said to be completely integrable in the sense of Liouville [29]

(i) if it is symplectic,
(ii) if there exist N functionally independent integrals $I_1(n), \ldots, I_N(n)$, which are mutually in involution with respect to the symplectic structure, that is, $\{I_m, I_r\} = 0$ for each pair (m, r), $m, r = 1, \ldots, N$, where

$$\{I_m, I_r\} = \sum_{i,j} \frac{\partial I_m}{\partial w_i} \Omega_{i,j} \frac{\partial I_r}{\partial w_j}. \tag{1.3}$$

The plan of the paper is as follows. In section 2 we explain how to construct four-dimensional QRT mappings with one or two rational integrals. In section 3, we discuss the question of integrability of the obtained four-dimensional mappings and summarize our results.

2. Construction of the rational integral for a fourth-order autonomous $O\Delta E$

Consider an autonomous fourth-order $O\Delta E$ having the form

$$w_{n+4} = F(w_n, w_{n+1}, w_{n+2}, w_{n+3}) \qquad \text{or} \qquad w_4 = F(w_0, w_1, w_2, w_3). \tag{2.1}$$

Hereafter we denote $w_n = w_0, w_{n+1} = w_1, \ldots, w_{n+N-1} = w_{N-1}$ unless otherwise specified. We look for a rational integral for (2.1) with the form

$$I(w_0, w_1, w_2, w_3) = \frac{P(w_0, w_1, w_2, w_3)}{Q(w_0, w_1, w_2, w_3)} = \frac{\sum_{j=1}^{3}\left[A_{1j}(w_1, w_2)w_3^2 + A_{2j}(w_1, w_2)w_3 + A_{3j}(w_1, w_2)\right]w_0^{3-j}}{\sum_{j=1}^{3}\left[a_{1j}(w_1, w_2)w_3^2 + a_{2j}(w_1, w_2)w_3 + a_{3j}(w_1, w_2)\right]w_0^{3-j}}, \tag{2.2}$$

where $A_{ij}(w_1, w_2)$ and $a_{ij}(w_1, w_2)$ are arbitrary unknown functions. We wish to mention that the case when $a_{1j} = 0, a_{3j} = 0, a_{21} = 0, a_{23} = 0, a_{22} = w_1 w_2$ has been considered in [24] and the case when $a_{1j} = 0, a_{2j} = 0, a_{31} = 0, a_{32} = 0, a_{33} = 1$ in [25].

The integral condition $I(w_n, w_{n+1}, w_{n+2}, w_{n+3}) = I(w_{n+1}, w_{n+2}, w_{n+3}, w_{n+4})$ leads to the quadratic equation in w_4 as

$$\begin{aligned}
&\left[\left(\sum_{j=1}^{3} A_{1j}(w_2, w_3)w_1^{3-j}\right) Q(w_0, w_1, w_2, w_3) - \left(\sum_{j=1}^{3} a_{1j}(w_2, w_3)w_1^{3-j}\right) P(w_0, w_1, w_2, w_3)\right] w_4^2 \\
&+ \left[\left(\sum_{j=1}^{3} A_{2j}(w_2, w_3)w_1^{3-j}\right) Q(w_0, w_1, w_2, w_3) - \left(\sum_{j=1}^{3} a_{2j}(w_2, w_3)w_1^{3-j}\right) P(w_0, w_1, w_2, w_3)\right] w_4 \\
&+ \left[\left(\sum_{j=1}^{3} A_{3j}(w_2, w_3)w_1^{3-j}\right) Q(w_0, w_1, w_2, w_3) - \left(\sum_{j=1}^{3} a_{3j}(w_2, w_3)w_1^{3-j}\right) P(w_0, w_1, w_2, w_3)\right] = 0.
\end{aligned} \tag{2.3}$$

Note that the above equation can be solved for w_4 in different ways. For example, if $A_{1i}(w_1, w_2) = a_{1i}(w_1, w_2) = 0, i = 1, 2, 3,$ and $A_{j1}(w_1, w_2) = a_{j1}(w_1, w_2) = 0,\ j = 2, 3,$ then we obtain the following $O\Delta E$:

$$w_4 = \frac{F_1 - w_0 F_2}{F_3 - w_0 F_4} \tag{2.4}$$

where each F_i is a function of w_1, w_2, w_3. Equation (2.4) can be viewed as a four-dimensional QRT mapping possessing one integral

$$I(w_0, w_1, w_2, w_3) = \frac{[A_{22}(w_1, w_2)w_3 + A_{32}(w_1, w_2)]\, w_0 + A_{23}(w_1, w_2)w_3 + A_{33}(w_1, w_2)}{[a_{22}(w_1, w_2)w_3 + a_{32}(w_1, w_2)]\, w_0 + a_{23}(w_1, w_2)w_3 + a_{33}(w_1, w_2)}, \tag{2.5}$$

where $A_{2j}(w_1, w_2)$, $A_{3j}(w_1, w_2)$, $a_{2j}(w_1, w_2)$ and $a_{3j}(w_1, w_2)$, $j = 2, 3$, are the arbitrary functions.

Here,

$$F_1 = \left(\sum_{j=2}^{3} a_{3j}(w_2, w_3)w_1^{3-j}\right)\left(\sum_{j=2}^{3} A_{j3}(w_1, w_2)w_3^{3-j}\right) - \left(\sum_{j=2}^{3} A_{3j}(w_2, w_3)w_1^{3-j}\right)\left(\sum_{j=2}^{3} a_{j3}(w_1, w_2)w_3^{3-j}\right),$$

$$F_2 = \left(\sum_{j=2}^{3} a_{j2}(w_1, w_2)w_3^{3-j}\right)\left(\sum_{j=2}^{3} A_{3j}(w_2, w_3)w_1^{3-j}\right) - \left(\sum_{j=2}^{3} A_{j2}(w_1, w_2)w_3^{3-j}\right)\left(\sum_{j=2}^{3} a_{3j}(w_2, w_3)w_1^{3-j}\right),$$

$$F_3 = \left(\sum_{j=2}^{3} a_{j3}(w_1, w_2)w_3^{3-j}\right)\left(\sum_{j=2}^{3} A_{2j}(w_2, w_3)w_1^{3-j}\right) - \left(\sum_{j=2}^{3} A_{j3}(w_1, w_2)w_3^{3-j}\right)\left(\sum_{j=2}^{3-j} a_{2j}(w_2, w_3)w_1^{3-j}\right),$$

$$F_4 = \left(\sum_{j=2}^{3} a_{2j}(w_2, w_3)w_1^{3-j}\right)\left(\sum_{j=2}^{3} A_{j2}(w_1, w_2)w_3^{3-j}\right) - \left(\sum_{j=2}^{3} A_{2j}(w_2, w_3)w_1^{3-j}\right)\left(\sum_{j=2}^{3} a_{j2}(w_1, w_2)w_3^{3-j}\right).$$

The integral condition equation (2.3) can also be solved for w_4 at least in two more distinct ways through factorization. For clarity we discuss them separately as cases 1 and 2.

Case 1. Equation (2.3) can be factored into

$$\left(w_4 - w_0\left[\frac{A_{31}(w_2, w_3)w_1^2 + A_{32}(w_2, w_3)w_1 + A_{33}(w_2, w_3)}{A_{13}(w_1, w_2)w_3^2 + A_{23}(w_1, w_2)w_3 + A_{33}(w_1, w_2)}\right]\right)\left(w_4 - \left[\frac{f_1 - w_0 f_2}{f_2 - w_0 f_3}\right] \times \left[\frac{A_{11}(w_1, w_2)w_3^2 + A_{21}(w_1, w_2)w_3 + A_{31}(w_1, w_2)}{A_{11}(w_2, w_3)w_1^2 + A_{12}(w_2, w_3)w_1 + A_{13}(w_2, w_3)}\right]\right) = 0 \tag{2.6}$$

provided the following conditions are satisfied:

$$\sum_{i=1}^{3} A_{2i}(w_2, w_3)w_1^{3-i} = \sum_{j=1}^{3} A_{j2}(w_1, w_2)w_3^{3-j}, \tag{2.7}$$

$$\sum_{i=1}^{3} a_{2i}(w_2, w_3)w_1^{3-i} = \sum_{j=1}^{3} a_{j2}(w_1, w_2)w_3^{3-j} \tag{2.8}$$

$$\begin{aligned}
&\frac{\left[A_{11}(w_1, w_2)w_3^2 + A_{21}(w_1, w_2)w_3 + A_{31}(w_1, w_2)\right]}{\left[A_{11}(w_2, w_3)w_1^2 + A_{12}(w_2, w_3)w_1 + A_{13}(w_2, w_3)\right]}\\
&\qquad = \frac{\left[A_{31}(w_2, w_3)w_1^2 + A_{32}(w_2, w_3)w_1 + A_{33}(w_2, w_3)\right]}{\left[A_{13}(w_1, w_2)w_3^2 + A_{23}(w_1, w_2)w_3 + A_{33}(w_1, w_2)\right]}\\
&\qquad = \frac{\left[a_{31}(w_2, w_3)w_1^2 + a_{32}(w_2, w_3)w_1 + a_{33}(w_2, w_3)\right]}{\left[a_{13}(w_1, w_2)w_3^2 + a_{23}(w_1, w_2)w_3 + a_{33}(w_1, w_2)\right]}\\
&\qquad = \frac{\left[a_{11}(w_1, w_2)w_3^2 + a_{21}(w_1, w_2)w_3 + a_{31}(w_1, w_2)\right]}{\left[a_{11}(w_2, w_3)w_1^2 + a_{12}(w_2, w_3)w_1 + a_{13}(w_2, w_3)\right]},
\end{aligned} \tag{2.9}$$

where

$$\left.\begin{aligned}
&f_1 = f_1(w_1, w_2, w_3) = A_2(w_1, w_2, w_3)a_3(w_1, w_2, w_3) - a_2(w_1, w_2, w_3)A_3(w_1, w_2, w_3),\\
&f_2 = f_2(w_1, w_2, w_3) = A_3(w_1, w_2, w_3)a_1(w_1, w_2, w_3) - a_3(w_1, w_2, w_3)A_1(w_1, w_2, w_3),\\
&f_3 = f_3(w_1, w_2, w_3) = A_1(w_1, w_2, w_3)a_2(w_1, w_2, w_3) - a_1(w_1, w_2, w_3)A_2(w_1, w_2, w_3),\\
&\left.\begin{aligned}
&A_i(w_1, w_2, w_3) = \textstyle\sum_{j=1}^{3} A_{ji}(w_1, w_2)w_3^{3-j},\\
&a_i(w_1, w_2, w_3) = \textstyle\sum_{j=1}^{3} a_{ji}(w_1, w_2)w_3^{3-j},
\end{aligned}\right\} \quad i = 1, 2, 3.
\end{aligned}\right\} \tag{2.10}$$

Now (2.6) can be rewritten as

$$w_4 = w_0 \left[\frac{A_{31}(w_2, w_3)w_1^2 + A_{32}(w_2, w_3)w_1 + A_{33}(w_2, w_3)}{A_{13}(w_1, w_2)w_3^2 + A_{23}(w_1, w_2)w_3 + A_{33}(w_1, w_2)}\right], \tag{2.11}$$

$$w_4 = \left[\frac{f_1 - w_0 f_2}{f_2 - w_0 f_3}\right]\left[\frac{A_{11}(w_1, w_2)w_3^2 + A_{21}(w_1, w_2)w_3 + A_{31}(w_1, w_2)}{A_{11}(w_2, w_3)w_1^2 + A_{12}(w_2, w_3)w_1 + A_{13}(w_2, w_3)}\right]. \tag{2.12}$$

Let us assume that

$$\frac{A_{11}(w_1, w_2)w_3^2 + A_{21}(w_1, w_2)w_3 + A_{31}(w_1, w_2)}{A_{11}(w_2, w_3)w_1^2 + A_{12}(w_2, w_3)w_1 + A_{13}(w_2, w_3)} = 1 \tag{2.13}$$

and so (2.12) reduces into a QRT-type mapping in four dimensions

$$w_4 = \left[\frac{f_1 - f_2 w_0}{f_2 - f_3 w_0}\right] \tag{2.14}$$

with 48 parameters admitting one integral (2.2) which is also cyclic invariant. Here

$$\left.\begin{aligned}
A_{11}(w_1, w_2) &= \left(\alpha_1 w_1^2 + \alpha_2 w_1 + \alpha_3\right)w_2^2 + \left(\alpha_2 w_1^2 + \alpha_4 w_1 + \alpha_5\right)w_2 + \alpha_3 w_1^2 + \alpha_6 w_1 + \alpha_7\\
A_{21}(w_1, w_2) &= \left(\alpha_2 w_1^2 + \alpha_8 w_1 + \alpha_9\right)w_2^2 + \left(\alpha_4 w_1^2 + \alpha_{10} w_1 + \alpha_{11}\right)w_2 + \alpha_5 w_1^2 + \alpha_{12} w_1 + \alpha_{13}\\
A_{31}(w_1, w_2) &= \left(\alpha_3 w_1^2 + \alpha_9 w_1 + \alpha_{14}\right)w_2^2 + \left(\alpha_6 w_1^2 + \alpha_{15} w_1 + \alpha_{16}\right)w_2 + \alpha_7 w_1^2 + \alpha_{17} w_1 + \alpha_{18}\\
A_{12}(w_1, w_2) &= \left(\alpha_2 w_1^2 + \alpha_4 w_1 + \alpha_6\right)w_2^2 + \left(\alpha_8 w_1^2 + \alpha_{10} w_1 + \alpha_{12}\right)w_2 + \alpha_9 w_1^2 + \alpha_{15} w_1 + \alpha_{17}\\
A_{22}(w_1, w_2) &= \left(\alpha_4 w_1^2 + \alpha_{10} w_1 + \alpha_{15}\right)w_2^2 + \left(\alpha_{10} w_1^2 + \alpha_{19} w_1 + \alpha_{20}\right)w_2 + \alpha_{11} w_1^2 + \alpha_{20} w_1 + \alpha_{21}\\
A_{32}(w_1, w_2) &= \left(\alpha_5 w_1^2 + \alpha_{11} w_1 + \alpha_{16}\right)w_2^2 + \left(\alpha_{12} w_1^2 + \alpha_{20} w_1 + \alpha_{22}\right)w_2 + \alpha_{13} w_1^2 + \alpha_{21} w_1 + \alpha_{23}\\
A_{13}(w_1, w_2) &= \left(\alpha_3 w_1^2 + \alpha_5 w_1 + \alpha_7\right)w_2^2 + \left(\alpha_9 w_1^2 + \alpha_{11} w_1 + \alpha_{13}\right)w_2 + \alpha_{14} w_1^2 + \alpha_{16} w_1 + \alpha_{18}\\
A_{23}(w_1, w_2) &= \left(\alpha_6 w_1^2 + \alpha_{12} w_1 + \alpha_{17}\right)w_2^2 + \left(\alpha_{15} w_1^2 + \alpha_{20} w_1 + \alpha_{21}\right)w_2 + \alpha_{16} w_1^2 + \alpha_{22} w_1 + \alpha_{23}\\
A_{33}(w_1, w_2) &= \left(\alpha_7 w_1^2 + \alpha_{13} w_1 + \alpha_{18}\right)w_2^2 + \left(\alpha_{17} w_1^2 + \alpha_{21} w_1 + \alpha_{23}\right)w_2 + \alpha_{18} w_1^2 + \alpha_{23} w_1 + \alpha_{24}.
\end{aligned}\right\} \tag{2.15}$$

The $a_{ij}(w_1, w_2)$ assumes the same form as $A_{ij}(w_1, w_2)$ replacing α_i's with β_i's in (2.15).

In order to construct a second integral for (2.14) we want to know under what conditions on the parameters, the asymmetric form of QRT mapping (2.4), becomes a symmetric form given in (2.14). As a consequence we obtain a set of two distinct QRT-type mappings with three parameters. Also both the mappings are symplectic and admit two independent integrals. The identified four-dimensional QRT-type mappings are as follows:

$$(i)\ w_4 = \frac{f_1 - f_2 w_0}{f_2 + f_1 w_0}, \tag{2.16}$$

where

$$\begin{aligned}
f_1 &= (w_3 w_1 - 1) f_{11} + (w_3 + w_1)(f_{12} - f_{13} - f_{14}),\\
f_2 &= (w_1 + w_3) f_{11} + (1 - w_1 w_3)(f_{12} - f_{13} - f_{14}),\\
f_{11} &= \gamma_1\gamma_3\big(2w_2 - w_1 + w_2^2 w_1 - w_3 + w_2^2 w_3 - 2w_1 w_2 w_3\big),\\
f_{12} &= \big(\gamma_1^2 + \gamma_2^2\big)(1 + w_1 w_2)(1 + w_2 w_3),\\
f_{13} &= \gamma_2\gamma_3(w_1 - w_3)\big(1 + w_2^2\big),\\
f_{14} &= \gamma_3^2 (w_2 - w_3)(w_2 - w_1).
\end{aligned}$$

The associated integrals $I_1(n)$, $I_2(n)$ read

$$I_1(n) = \frac{\gamma_1\gamma_3(w_3 - w_0)\tau_1 + (1 + w_0 w_3)\big[\big(\gamma_1^2 + \gamma_2^2\big)(1 + w_1 w_2) + \gamma_2\gamma_3\tau_2 - \gamma_3^2(w_2 - w_1)\big]}{\gamma_1(1 + w_1 w_2)(1 + w_0 w_3) - (w_0 - w_3)[\gamma_2(1 + w_1 w_2) + \gamma_3(w_2 - w_1)]} \tag{2.17}$$

$$I_2(n) = \frac{\tau_3\big(\gamma_1^2 + \gamma_2^2\big) + \tau_4\gamma_1\gamma_3 + \gamma_3^2\tau_6 + \gamma_2(\gamma_3\tau_5\tau_7 - \tau_8)}{\big(\gamma_1^2 + \gamma_2^2\big)\tau_9 - \tau_6\gamma_3^2 - \tau_7\tau_5\gamma_2\gamma_3 - \gamma_1(\tau_4\gamma_3 + \tau_8)}, \tag{2.18}$$

where

$$\begin{aligned}
\tau_1 &= w_1 w_2 - w_1 + w_2 + 1,\\
\tau_2 &= -w_1 w_2 - w_1 + w_2 - 1,\\
\tau_3 &= \big[\big(w_1^2 + 1 - w_1 w_2 + w_2^2\big)w_0^2 - w_2(1 + w_1 w_2)w_0 + \big(w_1^2 + 1\big)\big(1 + w_2^2\big)\big]w_3^2\\
&\qquad - (w_0 + w_2)(1 + w_0 w_1)(1 + w_1 w_2)w_3 + \big(w_1^2 + 1\big)\big(1 + w_2^2\big)w_0^2\\
&\qquad - w_1(1 + w_1 w_2)w_0 + w_2^2 + w_1^2 - w_1 w_2 + w_2^2 w_1^2,\\
\tau_4 &= \big[(-w_1 w_2 - 1)w_0^2 + \big(w_1 - w_2^2 w_1 + 2w_2 w_1^2\big)w_0 + w_1 w_2 - w_2^2\big]w_3^2\\
&\qquad + (1 - w_2 w_0)\big(\big(-2w_1 w_2 - 1 + w_1^2\big)w_0 - w_2 + w_2 w_1^2 + 2w_1\big)w_3\\
&\qquad \times \big(w_1 w_2 - w_1^2\big)w_0^2 + \big(-w_1 + 2w_2 + w_2^2 w_1\big)w_0 - w_2^2 w_1^2 - w_1 w_2,\\
\tau_5 &= (-w_1 w_0 - 1 + w_2 w_0 - w_2 w_1)w_3 - w_1 + w_0 w_1 w_2 + w_0 + w_2,\\
\tau_6 &= (w_1 - w_0)(w_2 - w_1)(w_2 - w_3)(w_0 - w_3),\\
\tau_7 &= (w_1 - w_3)(w_2 - w_0),\\
\tau_8 &= \big(1 + w_3^2\big)\big(1 + w_2^2\big)\big(1 + w_1^2\big)\big(1 + w_0^2\big),\\
\tau_9 &= (1 + w_2 w_3)(1 + w_1 w_2)(1 + w_0 w_3)(1 + w_0 w_1)
\end{aligned}$$

and γ_1, γ_2 and γ_3 are parameters. We mention here that the above four-dimensional mapping (2.16) is a measure preserving one with measure

$$\big[\big(\gamma_1^2 + \gamma_2^2\big)\tau_9 - \tau_6\gamma_3^2 - \tau_7\tau_5\gamma_2\gamma_3 - \gamma_1(\tau_4\gamma_3 + \tau_8)\big]^{-1}.$$

$$(ii)\ \ w_4 = \frac{f_1 - f_2 w_0}{f_2 - f_1 w_0}, \tag{2.19}$$

where

$$f_1 = (\gamma_1-\gamma_2)(w_1+w_3)(w_1-w_2)(w_3-w_2)+\gamma_3\big[\big((-1-w_3)w_2^2+2w_3^2w_2-w_3+1\big)w_1^2 + \big(\big(-1-w_3^2\big)w_2^2+4w_3w_2-1-w_3^2\big)w_1+\big(w_3^2-w_3\big)w_2^2+2w_2-w_3^2-w_3\big],$$

$$f_2 = (\gamma_2-\gamma_1)(w_2-w_3)(1+w_1w_3)(w_1-w_2)-\gamma_3\big[\big((1+w_3)w_2^2-2w_3w_2-w_3+1\big)w_1^2 + \big(\big(1-w_3^2+2w_3\big)w_2^2-2\big(1+w_3^2\big)w_2+2w_3-1+w_3^2\big)w_1+\big(w_3^2-w_3\big)w_2^2 - 2w_3w_2+w_3^2+w_3\big].$$

The associated integrals $I_1(n)$, $I_2(n)$ read

$$I_1(n) = \frac{\gamma_4(w_1-w_2)(1+w_0)(w_3-1)}{(w_1-w_2)\left[\gamma_1(w_3-w_0)+\gamma_2(w_0w_3-1)\right]-\gamma_3(w_0-1)(w_3+1)(w_2w_1-1)}, \tag{2.20}$$

$$I_2(n) = \frac{\gamma_5\tau_1+\gamma_4\gamma_3\tau_2+\gamma_4(\gamma_2-\gamma_1)\tau_3}{\gamma_4(\gamma_1-\gamma_2)\tau_3-\gamma_4\gamma_3\tau_2+\gamma_6\tau_1}, \tag{2.21}$$

where

$$\tau_1 = \big(w_0^2-1\big)\big(w_3^2-1\big)\big(w_2^2-1\big)\big(w_1^2-1\big),$$

$$\tau_3 = (w_3-w_0)(w_2-w_3)(w_1-w_0)(w_1-w_2),$$

$$\tau_2 = \big((w_2-1)(1+w_1)w_3^2+\big(w_1^2+w_2+w_1w_2(w_1-2w_2)-1\big)w_3 - (1+w_2)(w_1-1)w_1\big)w_0^2+\big(\big(1-w_2^2-2w_2w_1^2+w_1w_2^2+w_1\big)w_3^2 + \big(w_2^2+w_1^2+1-4w_1w_2+w_1^2w_2^2\big)w_3+w_1w_2^2+w_1^2w_2^2-w_1^2+w_1-2w_2\big)w_0 + w_2(1+w_2)(w_1-1)w_3^2+\big(-w_2^2w_1^2+w_2^2+w_2+w_2w_1^2-2w_1\big)w_3 - w_1w_2(w_2-1)(1+w_1)$$

and γ_1, γ_2 and γ_3 are parameters.

We would like to mention that the above four-dimensional mapping (2.19) is a measure preserving one with measure

$$[\gamma_4(\gamma_1-\gamma_2)\tau_3-\gamma_4\gamma_3\tau_2+\gamma_6\tau_1]^{-1}.$$

The associated symplectic structures of the above four-dimensional mappings (2.16) and (2.19) are given in the appendix.

We have identified another four-dimensional QRT-type map which is linearizable (globally) and admits one integral. The mapping and the integral read

$$(iii)\quad w_4 = \frac{2p(w_1,w_2,w_3)q(w_1,w_2,w_3)+[p(w_1,w_2,w_3)^2-q(w_1,w_2,w_3)^2]w_0}{[q(w_1,w_2,w_3)^2-p(w_1,w_2,w_3)^2]+2p(w_1,w_2,w_3)q(w_1,w_2,w_3)w_0} \tag{2.22}$$

where

$$p(w_1,w_2,w_3) = \lambda w_2\big(w_2^2-3\big)+2(w_1+w_3)(w_1w_3-1)\big(3w_2^2-1\big),$$

$$q(w_1,w_2,w_3) = 2(w_1+w_3)(w_1w_3-1)w_2\big(w_2^2-3\big)+\lambda\big(1-3w_2^2\big),$$

$$\lambda = (w_1w_3+w_3-1+w_1)(-w_3-w_1+w_1w_3-1).$$

The explicit form of the integral is

$$I(n) = \frac{A(w_3-w_0)+(w_1-w_2)\big(3w_1^2w_2^2-w_1^2+8w_1w_2+3-w_2^2\big)(1+w_0w_3)}{A(1+w_3w_0)-(w_1-w_2)\big(3w_1^2w_2^2-w_1^2+8w_1w_2+3-w_2^2\big)(w_3-w_0)} \tag{2.23}$$

$$A = (1 + w_1 w_2)\big(1 - 3w_1^2 + w_1^2 w_2^2 + 8w_1 w_2 - 3w_2^2\big).$$

As mentioned above (2.22) can be transformed into a linear fourth-order $O\Delta E$

$$\theta(n+4) - 4\theta(n+3) + 6\theta(n+2) - 4\theta(n+1) + \theta(n) = p\pi, \qquad \theta(n) = \arctan(w(n)), \tag{2.24}$$

and so the general solution of (2.22) is

$$w(n) = \tan\left(\frac{k\pi}{24}n^4 + c_1 n^3 + c_2 n^2 + c_3 n + c_4\right), \qquad k \in \mathbb{Z}, \tag{2.25}$$

where c_1, c_2, c_3, c_4 are arbitrary constants.

We would like to mention that the above four-dimensional mapping (2.22) is a measure preserving one with measure

$$\frac{\big(1 + w_1^2\big)^2\big(1 + w_2^2\big)^2}{\big[A(1 + w_3 w_0) - (w_1 - w_2)\big(3w_1^2 w_2^2 - w_1^2 + 8w_1 w_2 + 3 - w_2^2\big)(w_3 - w_0)\big]^2}.$$

Case 2. Equation (2.3) can be factored into

$$\left(w_4 - \frac{1}{w_0}\left[\frac{A_{13}(w_1, w_2)w_3^2 + A_{23}(w_1, w_2)w_3 + A_{33}(w_1, w_2)}{A_{11}(w_2, w_3)w_1^2 + A_{12}(w_2, w_3)w_1 + A_{13}(w_2, w_3)}\right]\right)\left(w_4 - \left[\frac{f_2 - f_3 w_0}{f_1 - f_2 w_0}\right] \times \left[\frac{A_{31}(w_2, w_3)w_1^2 + A_{32}(w_2, w_3)w_1 + A_{33}(w_2, w_3)}{A_{11}(w_1, w_2)w_3^2 + A_{21}(w_1, w_2)w_3 + A_{31}(w_1, w_2)}\right]\right) = 0, \tag{2.26}$$

provided in addition to (2.7) and (2.8) the following conditions are satisfied:

$$\begin{aligned}
&\frac{\big[A_{11}(w_2, w_3)w_1^2 + A_{12}(w_2, w_3)w_1 + A_{13}(w_2, w_3)\big]}{\big[A_{13}(w_1, w_2)w_3^2 + A_{23}(w_1, w_2)w_3 + A_{33}(w_1, w_2)\big]} \\
&\quad = \frac{\big[a_{11}(w_2, w_3)w_1^2 + a_{12}(w_2, w_3)w_1 + a_{13}(w_2, w_3)\big]}{\big[a_{13}(w_1, w_2)w_3^2 + a_{23}(w_1, w_2)w_3 + a_{33}(w_1, w_2)\big]} \\
&\quad = \frac{\big[A_{11}(w_1, w_2)w_3^2 + A_{21}(w_1, w_2)w_3 + A_{31}(w_1, w_2)\big]}{\big[A_{31}(w_2, w_3)w_1^2 + A_{32}(w_2, w_3)w_1 + A_{33}(w_2, w_3)\big]} \\
&\quad = \frac{\big[a_{11}(w_1, w_2)w_3^2 + a_{21}(w_1, w_2)w_3 + a_{31}(w_1, w_2)\big]}{\big[a_{31}(w_2, w_3)w_1^2 + a_{32}(w_2, w_3)w_1 + a_{33}(w_2, w_3)\big]},
\end{aligned} \tag{2.27}$$

where $f_i(w_1, w_2, w_3), i = 1, 2, 3$, are as given in (2.10). Obviously (2.26) can be rewritten as

$$w_4 = \frac{1}{w_0}\left[\frac{A_{13}(w_1, w_2)w_3^2 + A_{23}(w_1, w_2)w_3 + A_{33}(w_1, w_2)}{A_{11}(w_2, w_3)w_1^2 + A_{12}(w_2, w_3)w_1 + A_{13}(w_2, w_3)}\right], \tag{2.28}$$

$$w_4 = \left[\frac{f_2 - f_3 w_0}{f_1 - f_2 w_0}\right]\left[\frac{A_{31}(w_2, w_3)w_1^2 + A_{32}(w_2, w_3)w_1 + A_{33}(w_2, w_3)}{A_{11}(w_1, w_2)w_3^2 + A_{21}(w_1, w_2)w_3 + A_{31}(w_1, w_2)}\right]. \tag{2.29}$$

As in case 1 we assume that

$$\left[\frac{A_{31}(w_2, w_3)w_1^2 + A_{32}(w_2, w_3)w_1 + A_{33}(w_2, w_3)}{A_{11}(w_1, w_2)w_3^2 + A_{21}(w_1, w_2)w_3 + A_{31}(w_1, w_2)}\right] = 1, \tag{2.30}$$

and so from (2.29) we obtain a QRT-type mapping in four dimensions

$$w_4 = \left[\frac{f_2 - f_3 w_0}{f_1 - f_2 w_0}\right] \tag{2.31}$$

with 22 parameters admitting one integral (2.2) where

$$\left.\begin{aligned}
A_{11}(w_1, w_2) &= \left(\alpha_1 w_1^2 + \alpha_2 w_1 + \alpha_3\right) w_2^2 + \left(\alpha_4 w_1^2 + \alpha_5 w_1 + \alpha_6\right) w_2 + \alpha_3 w_1^2 + \alpha_6 w_1 + \alpha_3 \\
A_{21}(w_1, w_2) &= \left(\alpha_7 w_1^2 + \alpha_8 w_1 + \alpha_6\right) w_2^2 + \left(\alpha_9 w_1^2 + \alpha_{10} w_1 + \alpha_5\right) w_2 + \alpha_2 w_1^2 + \alpha_8 w_1 + \alpha_4 \\
A_{31}(w_1, w_2) &= \left(\alpha_1 w_1^2 + \alpha_4 w_1 + \alpha_3\right) w_2^2 + \left(\alpha_7 w_1^2 + \alpha_9 w_1 + \alpha_2\right) w_2 + \alpha_1 w_1^2 + \alpha_7 w_1 + \alpha_1 \\
A_{12}(w_1, w_2) &= \left(\alpha_7 w_1^2 + \alpha_9 w_1 + \alpha_4\right) w_2^2 + \left(\alpha_8 w_1^2 + \alpha_{10} w_1 + \alpha_8\right) w_2 + \alpha_6 w_1^2 + \alpha_5 w_1 + \alpha_2 \\
A_{22}(w_1, w_2) &= \left(\alpha_9 w_1^2 + \alpha_{10} w_1 + \alpha_5\right) w_2^2 + \left(\alpha_{10} w_1^2 + \alpha_{11} w_1 + \alpha_{10}\right) w_2 + \alpha_5 w_1^2 + \alpha_{10} w_1 + \alpha_9 \\
A_{32}(w_1, w_2) &= \left(\alpha_2 w_1^2 + \alpha_5 w_1 + \alpha_6\right) w_2^2 + \left(\alpha_8 w_1^2 + \alpha_{10} w_1 + \alpha_8\right) w_2 + \alpha_4 w_1^2 + \alpha_9 w_1 + \alpha_7 \\
A_{13}(w_1, w_2) &= \left(\alpha_1 w_1^2 + \alpha_7 w_1 + \alpha_1\right) w_2^2 + \left(\alpha_2 w_1^2 + \alpha_9 w_1 + \alpha_7\right) w_2 + \alpha_3 w_1^2 + \alpha_4 w_1 + \alpha_1 \\
A_{23}(w_1, w_2) &= \left(\alpha_4 w_1^2 + \alpha_8 w_1 + \alpha_2\right) w_2^2 + \left(\alpha_5 w_1^2 + \alpha_{10} w_1 + \alpha_9\right) w_2 + \alpha_6 w_1^2 + \alpha_8 w_1 + \alpha_7 \\
A_{33}(w_1, w_2) &= \left(\alpha_3 w_1^2 + \alpha_6 w_1 + \alpha_3\right) w_2^2 + \left(\alpha_6 w_1^2 + \alpha_5 w_1 + \alpha_4\right) w_2 + \alpha_3 w_1^2 + \alpha_2 w_1 + \alpha_1.
\end{aligned}\right\} \tag{2.32}$$

$a_{ij}(w_1, w_2)$ assumes the same form as $A_{ij}(w_1, w_2)$ replacing α_i's with β_i's in (2.32).

Proceeding along the lines of case 1 we find a fourth-order QRT-type mapping admitting two independent integrals. The explicit form of the mapping

$$w_4 = \frac{f_2 + f_1 w_0}{f_1 - f_2 w_0} \tag{2.33}$$

$$f_2 = \alpha_1^2 (1 + w_1 w_2)(1 + w_2 w_3)(w_3 - w_1) + \alpha_1 \alpha_2 \left(1 + w_2^2\right)\left(w_3^2 - w_1^2\right)$$

$$\begin{aligned}
f_1 = \alpha_1^2 (1 + w_1 w_2)(1 + w_2 w_3)(1 + w_1 w_3) + \alpha_2^2 (1 + w_1^2)\left(1 + w_2^2\right)\left(1 + w_3^2\right) \\
+ \alpha_1 \alpha_2 \left(w_1^2 w_2^2 w_3 + w_2^2 w_3^2 w_1 + 2 w_3^2 w_1^2 w_2 + w_2^2 w_3 + 2 w_3^2 w_2 + 2 w_2 w_1^2 + w_1^2 w_3 \right. \\
\left. + w_3^2 w_1 + w_2^2 w_1 + 2 w_2 + w_1 + w_3\right)
\end{aligned}$$

and its integrals $I_1(n)$ and $I_2(n)$ are

$$I_1(n) = \frac{P(n)}{Q(n)} \tag{2.34}$$

$$\begin{aligned}
P(n) = \alpha_2 \alpha_1 (-1 + w_0 w_3)(w_1 w_2 + 1) + \alpha_2^2 [(w_3 w_1 + w_2 w_3 + w_1 w_2 - 1) w_0 \\
- w_1 - w_2 + w_1 w_2 w_3 - w_3]
\end{aligned}$$

$$\begin{aligned}
Q(n) = \alpha_1 (-1 + w_0 w_3 - w_3 - w_0)(w_1 w_2 + 1) + \alpha_2 [((-1 + w_1 w_2 + w_1 + w_2) w_3 + w_1 w_2 \\
- 1 - w_1 - w_2) w_0 + (w_1 w_2 - 1 - w_1 - w_2) w_3 - w_1 - w_2 - w_1 w_2 + 1]
\end{aligned}$$

$$I_2(n) = \frac{\tau_1 \alpha_1^2 + \tau_2 \alpha_2 \alpha_1 + \tau_3 \alpha_2^2 + \tau_4}{-\tau_1 \alpha_1^2 - \tau_2 \alpha_2 \alpha_1 - \tau_3 \alpha_2^2 + \tau_4} \tag{2.35}$$

$$\tau_1 = (w_0 + w_3)(w_2 w_3 + 1)(w_1 w_0 + 1)(w_1 w_2 + 1)$$

$$\begin{aligned}
\tau_2 = \left[\left(w_1 w_2 - w_1^2 - w_2^2 - 1\right) w_3^2 + \left(w_2 + 2 w_2^2 w_1 + 2 w_1 + w_2 w_1^2\right) w_3 - w_2(-w_1 + w_2)\right] w_0^2 \\
+ \left[\left(2 w_2 + w_2^2 w_1 + 2 w_2 w_1^2 + w_1\right) w_3^2 + \left(w_1^2 + 1\right)\left(1 + w_2^2\right) w_3 + 2 w_2 + w_2^2 w_1 \right. \\
\left. + 2 w_2 w_1^2 + w_1\right] w_0 + w_1(-w_1 + w_2) w_3^2 + \left(w_2 + 2 w_2^2 w_1 + 2 w_1 + w_2 w_1^2\right) w_3 \\
- w_1^2 + w_1 w_2 - w_2^2 w_1^2 - w_2^2
\end{aligned}$$

$$\tau_3 = \left[(w_1+w_2)(1+w_1w_2)w_3^2 + \left(w_1^2+1\right)\left(1+w_2^2\right)w_3 + (w_1+w_2)(1+w_1w_2)\right]w_0^2$$
$$+\left[\left(w_1^2+1\right)\left(1+w_2^2\right)w_3^2 + \left(w_1^2+1\right)\left(1+w_2^2\right)\right]w_0 + (w_1+w_2)(1+w_1w_2)w_3^2$$
$$+\left(w_1^2+1\right)\left(1+w_2^2\right)w_3 + (w_1+w_2)(1+w_1w_2)$$
$$\tau_4 = \left(1+w_0^2\right)\left(1+w_1^2\right)\left(1+w_2^2\right)\left(1+w_3^2\right).$$

The above four-dimensional mapping (2.33) is a measure preserving one with measure

$$\left[-\tau_1\alpha_1^2 - \tau_2\alpha_2\alpha_1 - \tau_3\alpha_2^2 + \tau_4\right]^{-1}.$$

3. Summary

In this paper we identify four distinct cases, namely (2.16), (2.19), (2.22) and (2.33) of four-dimensional analogs of two-dimensional QRT mapping which are measure preserving. Furthermore the mappings given in (2.16) and (2.19) are symplectic and possess two independent integrals which are in involution with respect to their symplectic structures and hence they are integrable in the sense of Liouville. Next, the four-dimensional mapping (2.22) admits one integral but is linearizable globally and hence it is integrable. Even though the mapping given in (2.33) is measure preserving and admits two independent integrals, it lacks the symplectic structure at the moment and hence it is not clear whether it is integrable or not. The integrability of (2.33) can be established, hopefully, by other means. We wish to mention that our search for four-dimensional analogs of two-dimensional integrable mappings is not exhaustive.

Acknowledgments

The authors thank the anonymous referees for their valuable and critical comments. The work of RS forms part of the research project funded by the Council of Scientific and Industrial Research (CSIR), New Delhi. Ms Uma Maheswari wishes to thank the University Grants Commission, New Delhi, for providing financial support in the form of Senior Research Fellow.

Appendix. Construction of symplectic structures for four-dimensional mappings

Consider an autonomous fourth-order $O\Delta E$

$$w_{n+4} = F(w_n, w_{n+1}, w_{n+2}, w_{n+3}), \tag{A.1}$$

which can be rewritten as a system of first-order $O\Delta E$s

$$\left.\begin{aligned} w_{i+1} &= x_i \\ x_{i+1} &= y_i \\ y_{i+1} &= z_i \\ z_{i+1} = w_{i+4} &= F(w_i, w_{i+1}, w_{i+2}, w_{i+3}) \end{aligned}\right\}. \tag{A.2}$$

Given an Nth ($N \geqslant 2$)-order mapping it is not clear what form of symplectic structure one should look for. Let us assume that (A.2) admits a symplectic structure $\Omega(n) = \Omega(w_n, w_{n+1}, w_{n+2}, w_{n+3})$ having the form

$$\Omega(n) = \begin{pmatrix} 0 & \sigma_1(n) & \sigma_2(n) & \sigma_3(n) \\ -\sigma_1(n) & 0 & \sigma_4(n) & \sigma_5(n) \\ -\sigma_2(n) & -\sigma_4(n) & 0 & \sigma_6(n) \\ -\sigma_3(n) & -\sigma_5(n) & -\sigma_6(n) & 0 \end{pmatrix} \tag{A.3}$$

which is obviously an anti-symmetric one. The mapping (A.2) is symplectic if it satisfies the condition

$$J(n)\Omega(n)J(n)^T = \Omega(n+1), \tag{A.4}$$

where $J(n)$ is the Jacobian of the mapping (A.2) given by

$$J(n) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\partial F}{\partial w_n} & \frac{\partial F}{\partial w_{n+1}} & \frac{\partial F}{\partial w_{n+2}} & \frac{\partial F}{\partial w_{n+3}} \end{pmatrix}. \tag{A.5}$$

Equating the entries from LHS to RHS in the matrix equation (A.4) we obtain the following:

$$\left.\begin{aligned} \sigma_1(n+1) &= \sigma_4(n) \\ \sigma_2(n+1) &= \sigma_5(n) \\ \sigma_4(n+1) &= \sigma_6(n) \\ \sigma_3(n+1) &= -\sigma_1(n)\tfrac{\partial F}{\partial w_n} + \sigma_1(n+1)\tfrac{\partial F}{\partial w_{n+2}} + \sigma_2(n+1)\tfrac{\partial F}{\partial w_{n+3}} \\ \sigma_5(n+1) &= -\sigma_2(n)\tfrac{\partial F}{\partial w_n} - \sigma_1(n+1)\tfrac{\partial F}{\partial w_{n+1}} + \sigma_4(n+1)\tfrac{\partial F}{\partial w_{n+3}} \\ \sigma_6(n+1) &= -\sigma_3(n)\tfrac{\partial F}{\partial w_n} - \sigma_2(n+1)\tfrac{\partial F}{\partial w_{n+1}} - \sigma_4(n+1)\tfrac{\partial F}{\partial w_{n+3}} \end{aligned}\right\}. \tag{A.6}$$

Equation (A.6) cannot be solved in general. However, there exists a solution for (A.6) at least for two of the five identified mappings given in (2.16) and (2.19). The explicit forms of $\Omega(n)$ are given below.

Symplectic structure for $O\Delta E$ (2.16)

For the four-dimensional mapping given in (2.16), $\Omega(n)$ satisfying (A.6) is

$$\Omega(n) = \begin{pmatrix} 0 & 0 & \sigma_2(n) & \sigma_3(n) \\ 0 & 0 & 0 & \sigma_5(n) \\ -\sigma_2(n) & 0 & 0 & 0 \\ -\sigma_3(n) & -\sigma_5(n) & 0 & 0 \end{pmatrix}$$

where

$$\sigma_2(n) = k\left(1+w_n^2\right)\left(1+w_{n+2}^2\right), \qquad \sigma_5(n) = k\left(1+w_{n+1}^2\right)\left(1+w_{n+3}^2\right)$$

$$\sigma_3(n) = k\frac{\left(1+w_n^2\right)\left(1+w_{n+3}^2\right)\left[\lambda(n)+\gamma_1\gamma_3\left(1+w_{n+1}^2\right)\left(1+w_{n+2}^2\right)\right]}{\lambda(n)}$$

$$\lambda(n) = \left(\gamma_1^2+\gamma_2^2\right)(1+w_{n+1}w_{n+2})^2 + 2\gamma_2\gamma_3(w_{n+2}-w_{n+1})(1+w_{n+1}w_{n+2}) + \gamma_3^2(w_{n+2}-w_{n+1})^2$$

where k is an arbitrary constant.

Symplectic structure for $O\Delta E$ (2.19)

For the four-dimensional mapping given in (2.19), $\Omega(n)$ satisfying (A.6) is given by

$$\Omega(n) = \begin{pmatrix} 0 & 0 & \sigma_2(n) & \sigma_3(n) \\ 0 & 0 & 0 & \sigma_5(n) \\ -\sigma_2(n) & 0 & 0 & 0 \\ -\sigma_3(n) & -\sigma_5(n) & 0 & 0 \end{pmatrix}$$

where

$$\sigma_2(n) = k\big(w_n^2 - 1\big)\big(w_{n+2}^2 - 1\big), \qquad \sigma_5(n) = k\big(w_{n+1}^2 - 1\big)\big(w_{n+3}^2 - 1\big)$$

$$\sigma_3(n) = k\frac{\lambda_3(n)\big(w_n^2 - 1\big)\big(w_{n+3}^2 - 1\big)}{\lambda_1(n)(w_{n+1} - w_{n+2})}$$

$$\lambda_1(n) = (\gamma_2 - \gamma_1)(w_{n+2} - w_{n+1}) + 2\gamma_3(w_{n+1}w_{n+2} - 1)$$

$$\lambda_3(n) = (\gamma_1 - \gamma_2)(w_{n+1} - w_{n+2})^2 + \gamma_3\big(2(w_2 - w_1)(1 - w_1w_2) + w_1^2w_2^2 - w_1^2 - w_2^2 + 1\big)$$

where k is an arbitrary constant.

References

[1] Ablowitz M J and Ladik J F 1976 A nonlinear difference scheme and inverse scattering *Stud. Appl. Math.* **55** 213–29

[2] Ablowitz M J and Ladik J F 1976 Nonlinear differential-difference equations and Fourier analysis. *J. Math. Phys.* **17** 1011–8

[3] Adler V E 2006 On a class of third order mappings with two rational invariants arXiv:nlin/0606056v1

[4] Arnold V I 1978 *Mathematical Methods of Classical Mechanics* (Berlin: Springer)

[5] Bruschi M, Ragnisco O, Santini P M and Gui Zhang Tu 1991 Integrable symplectic maps *Physica* D **49** 273–94

[6] Capel H W and Sahadevan R 2001 A new family of four-dimensional symplectic and integrable mappings *Physica* A **289** 86–106

[7] Grammaticos B, Ramani A and Papageorgiou V G 1991 Do integrable mappings have the Painleve property? *Phys. Rev. Lett.* **67** 1825–8

[8] Grammaticos B, Kosmann-Schwarzbach Y and Tamizhmani T 2004 *Discrete Integrable Systems* (Berlin: Springer)

[9] Grammaticos B and Ramani A 2007 Integrable mappings with transcendental invariants *Commun. Nonlinear Sci. Numer. Simul.* **12** 350–6

[10] Hirota R, Kimura K and Yahagi H 2001 How to find the conserved quantities of nonlinear discrete equations *J. Phys. A: Math. Gen.* **34** 10377–86

[11] Hirota R and Yahagi H 2002 "Recurrence Equations", an integrable system *J. Phys. Soc. Japan* **71** 2867–72

[12] Iatrou A 2003 Three dimensional integrable mappings arXiv:nlin/0306052v1

[13] Iatrou A 2003 Higher dimensional integrable mappings *Physica* D **179** 229–53

[14] McMillan E C 1971 *Topics in Physics* ed W E Brittin and H Odabasi (Boulder: Colorado University Press) p 219

[15] Matsukidaira J and Takahashi D 2006 Third-order integrable difference equations generated by a pair of second-order equations *J. Phys. A:Math. Gen.* **39** 1151–61

[16] Papageorgiou V G, Nighoff F W and Capel H W 1990 Integrable mappings and nonlinear integrable lattice equations *Phys. Lett.* A **147** 106–14

[17] Peter H, Rojas O and Quispel G R W 2007 Closed-form expressions for integrals of MKdV and Sine-Gordon maps *J. Phys. A: Math. Theor.* **40** 12789–98

[18] Quispel G R W, Capel H W, Papageorgiou V G and Nijhoff F W 1991 Integrable mappings derived from soliton equations *Physica* A **173** 243–66

[19] Quispel G R W, Capel H W and Roberts J A G 2005 Duality for discrete integrable systems *J. Phys. A: Math. Gen.* **38** 3965–80

[20] Quispel G R W, Roberts J A G and Thompson C J 1988 Integrable mappings and soliton equations *Phys. Lett.* A **126** 419–21

[21] Quispel G R W, Roberts J A G and Thompson C J 1989 Integrable mappings and soliton equations II *Physica* D **34** 183–92

[22] Ramani A, Grammaticos B and Lafortune S 2002 The discrete Chazy III system of Labtunie–Conte is not integrable *J. Phys. A: Math. Gen.* **35** 7943–6

[23] Roberts J A G and Quispel G R W 2006 Creating and relating 3-dimensional integrable maps *J. Phys. A: Math. Gen.* **39** L605–15

[24] Sahadevan R and Uma Maheswari C 2008 Direct method to construct integrals for *N*th-order autonomous ordinary difference equations *Proc. R. Soc. A.* **464** 341–64

[25] Sahadevan R and Uma Maheswari C 2008 Polynomial integrals for third- and fourth-order ordinary difference equations *J. Nonlinear Math. Phys.* **15** 299–315

[26] Sahadevan R and Uma Maheswari C 2009 Third order difference equations with two rational integrals *J. Phys. A: Math. Theor.* **42** 454017

[27] Sahadevan R and Balakrishnan S 2009 Complete integrability of two coupled discrete modified Korteweg-de Vries equations. *J. Phys. A: Math. Theor.* **42** 415208

[28] Tsuda 2004 Integrable mappings via rational elliptic surfaces *J. Phys. A: Math. Gen.* **37** 2721–30

[29] Veselov A P 1991 Integrable maps *Russian Math. Surveys* **46** 1–51